ON THE SPECTRAL LINES OF A PULSATING STAR

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The hypothesis that the periodic variations in light and spectrum of a Cepheid variable are due to radial expansion and contraction of a single stellar body demands a behavior in the periodic oscillation of the spectral lines which is decidedly different from the behavior of the absorption lines of an ordinary spectroscopic binary star. In the latter case the line-shifts show, in accordance with the Doppler effect, the orbital motion of the star as a whole; in the former case the displacements of the spectral lines depend on the radial velocity of the part of the stellar disk from which the light emanates. The radial velocity due to a spherical pulsation diminishes from a maximum value at the center of the stellar disk to zero at the limb; the shifts, therefore, are not true bodily displacements of the normal spectral line, as in the case of orbital motion, but represent a broadening toward the violet while the star expands and toward the red while it contracts. Obviously a lack of symmetry as well as a widening is produced; but if the total displacement, due to radial movement of the absorbing layers, is of the same magnitude as the inherent width of the lines, it is clear that the asymmetrical broadening may not be readily noticed or measured. The object of this note is to examine in some detail the character of the spectral lines of a pulsating star.

For a star of unit radius undergoing alternate expansion and contraction symmetrically about the center, the radial velocity, in terms of its maximum value, for any point on the visible hemisphere is $v = \sqrt{1 - r^2}$, where r is the distance from the center of the apparent disk. Hence

$$r = \sqrt{1 - v^2} \tag{1}$$

The usual law of darkening, which has been found satisfactory in the case of the sun and eclipsing stars[1] and has subsequently been derived by Jeans[2] in his theory of stellar photospheres, may be written

$$\frac{J}{J_0} = 1 - x + x\sqrt{1 - r^2}$$

where J/J_0 is the surface intensity at the distance r from the center in terms of the central intensity, and x is the so-called darkening

coefficient. Then the total light of a star is given by

$$L = J_0 \int_0^{2\pi} \int_0^1 (1 - x + x\sqrt{1 - r^2})\, rdrd\theta$$

Introducing (1) and integrating once

$$L = 2\pi\, J_0 \left[\int_0^1 v\,(1 - x)\, dv + \int_0^1 v^2 x dv \right] \tag{2}$$

Suppose that from any given element of surface the width of the absorption lines is very small compared with the displacement due to radial motion of the element; and let the amount by which the continuous spectrum is decreased by a given absorption line be $L' = L$ *times* a constant. The equation of the intensity curve of a line in the spectrum of the whole surface would then be $\frac{I}{I_0} = \frac{dL'}{d\lambda}$ which may be written $\frac{dL'}{dv}$ because of the essentially linear relation (throughout very short intervals of the spectrum) between wave-length, λ, and velocity; and hence, from (2),

$$\frac{I}{I_0} = 2\pi\,[\,v\,(1 - x) + v^2 x] \tag{3}$$

where I_0 is the intensity for maximum absorption ($v = 1$) divided by 2π and is independent of the degree of darkening.

For a uniform disk $x = 0$ and

$$\frac{I}{I_0} = 2\pi v \tag{4}$$

while for one darkened to zero at the limb, $x = 1$, and

$$\frac{I}{I_0} = 2\pi v^2 \tag{5}$$

These intensity curves are plotted in figure 1. For a star of the spectral character of the sun, x is $\frac{3}{4}$ in the neighborhood of $\lambda = 4400$ A. Therefore

$$\frac{I}{I_0} = \frac{\pi}{2}\,(v + 3v^2)$$

and the intensity curve lies between those for uniform and completely darkened disks, but much nearer the latter, as shown by the broken curve in figure 1.

It is obvious from the above discussion that darkening at the limb will aid in concealing whatever asymmetry there may be in the spectral

lines of a pulsating star. From the eclipsing stars that most resemble the typical Cepheid variable stars in density, spectral type, and absolute luminosity, there is evidence in certain cases that the light diminishes more rapidly toward the limb than required by the above law. On the other hand, however, the relative intensity in the violet region of the continuous spectrum of Cepheids and other giant F-type stars

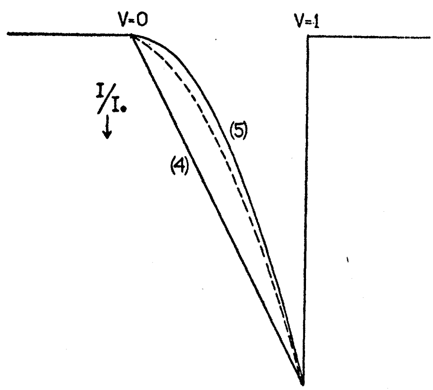

FIG. 1

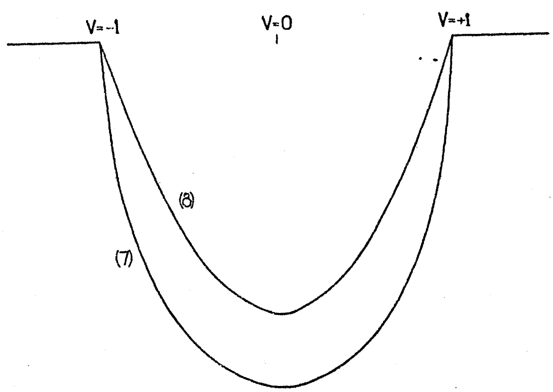

FIG. 2

may be urged against the assumption of excessive darkening. In any case, in view of the existence of large numbers of strong absorption lines throughout the spectra, it may be accepted that Cepheid variables are considerably darkened at the limb. For them the coefficient x may be reasonably assumed to lie between one-half and unity.

The preliminary assumption that spectral lines are inherently infinitesimal in width is of course untrue. Even if the radiation concerned

were strictly monochromatic the spectral lines would have a finite width since the resolving power of a spectrograph is not infinite. As is well known, however, the light giving rise to an absorption line is by no means monochromatic. Its distribution may be represented satisfactorily by

$$\frac{I'}{I'_0} = e^{-k^2\lambda^2} = e^{-k^2v^2/c^2} = e^{-k^2(v-v_n)^2} \qquad (6)$$

where v_n is the value of v for intensity I', corresponding to maximum absorption. The form of the resulting line, as reproduced by the spectrograph, has been discussed in detail by Wadsworth.[3]

Rotation of the star will also contribute to the widening of spectral lines. Since v, the projection of the rotational velocity in the line of sight, is numerically equivalent to the distance on the stellar disk from the axis of rotation, we may write $r^2 = v^2 + y^2$. Therefore the total light of the star is

$$L = \int\int J\, dv\, dy = J_0 \int_{-\sqrt{1-v^2}}^{+\sqrt{1-v^2}} \int_{-1}^{+1} [\,(1 - x) + x\sqrt{1 - v^2 - y^2}]\, dv\, dy$$

and

$$\frac{I}{I_R} = 2\,(1 - x)\sqrt{1 - v^2} + \frac{\pi x}{2}\,(1 - v^2)$$

In this case I_R is a function of the degree of darkening, being the intensity for maximum absorption (when $v = 0$) divided by $2\,(1 - x) + \frac{\pi x}{2}$.

Then for $x = 0$

$$\frac{I}{I_R} = 2\sqrt{1 - v^2} \qquad (7)$$

and for $x = 1$

$$\frac{I}{I_R} = \frac{\pi}{2}\,(1 - v^2) \qquad (8)$$

For comparison with figure 1, the intensity curves of the absorption lines for a rotating star (assuming monochromatic radiation and infinite resolving power) are shown in figure 2.

It can be shown without difficulty that if for an average Cepheid the period of rotation is less than two months the widening of the lines due to rotation will be decidedly greater than the widening due to pulsation.

Obviously, then, rotation of the star, resolving power, and the inherently finite width will all affect the appearance of the absorption lines, independently of changes due to pulsation. If the resolution is

not too small, however, and the widening due to rotation is not too large compared with the normal width of the line, these three factors will not appreciably affect the *form* of the line given by (6). The narrowest lines observed in stellar spectra are in width rarely less than 0.2A, which is of the same order as the semi-amplitude of the shift for a typical Cepheid variable.

The true form of a line in the spectrum of a pulsating star is therefore not as given in figure 1, but may be approximated by integrating (6) over the whole surface of the star. Any given incremental annulus on the stellar disk has a velocity v_n and contributes an incremental spectral line whose intensity curve is given by (6):

$$I' = I'_0\, e^{-K^2(v-v_n)^2}$$

The intensity of this line for maximum absorption, I'_0, occurs for $v = v_n$; and hence from (3),

$$I'_0 = I_n = 2\pi\, I_0\, [v_n\, (1 - x) + v_n^2\, x] \tag{9}$$

Substituting (9) in (6)

$$I' = 2\pi\, I_0\, [v_n\, (1 - x) + v_n^2\, x]\, e^{-K^2(v-v_n)^2}$$

and summing up these incremental lines for all chosen values of v_n from 0 to 1, we obtain, as closely as we care to compute, the intensity curve from the whole surface:

$$I'' = F\,(v, x) = \sum_{v_n=0}^{1} I'$$

Writing $c\lambda = v$ from (6), and $I''_0 = 2\pi I_0$, we have, in the limit, for the intensity of the built-up absorption line at any point λ,

$$\frac{I''}{I''_0} = f\,(\lambda, x) = \int_0^1 [\, v_n\, (1 - x) + v_n^2\, x]\, e^{-K^2(c\lambda-v_n)^2}\, dv_n \tag{10}$$

Equation (10) has been integrated mechanically for different values of x and different ratios of line width to displacements; figures 3a and 3b show the resulting intensity curves for line widths of about 0.4A and 0.2A, respectively, adopting in both cases a radial velocity that would yield a measured displacement of about 0.3A. From these curves we obtain the results of the following table:

	WIDTH OF UNDISPLACED LINE	WIDTH OF DISPLACED LINE		MEASURED DISPLACEMENT		ACTUAL DISPLACEMENT AT CENTER OF DISK
		$x=\frac{1}{2}$	$x=1$	$x=\frac{1}{2}$	$x=1$	
From figure 3a..........	0.34A	0.40A	0.38A	0.30A	0.32A	0.40A
From figure 3b..........	0.17A	0.23A	0.22A	0.32A	0.33A	0.40A

The last three columns of the table show that, if the pulsation hypothesis is correct, the measured range in velocity is about 75% of the actual variation at the center of the disk.

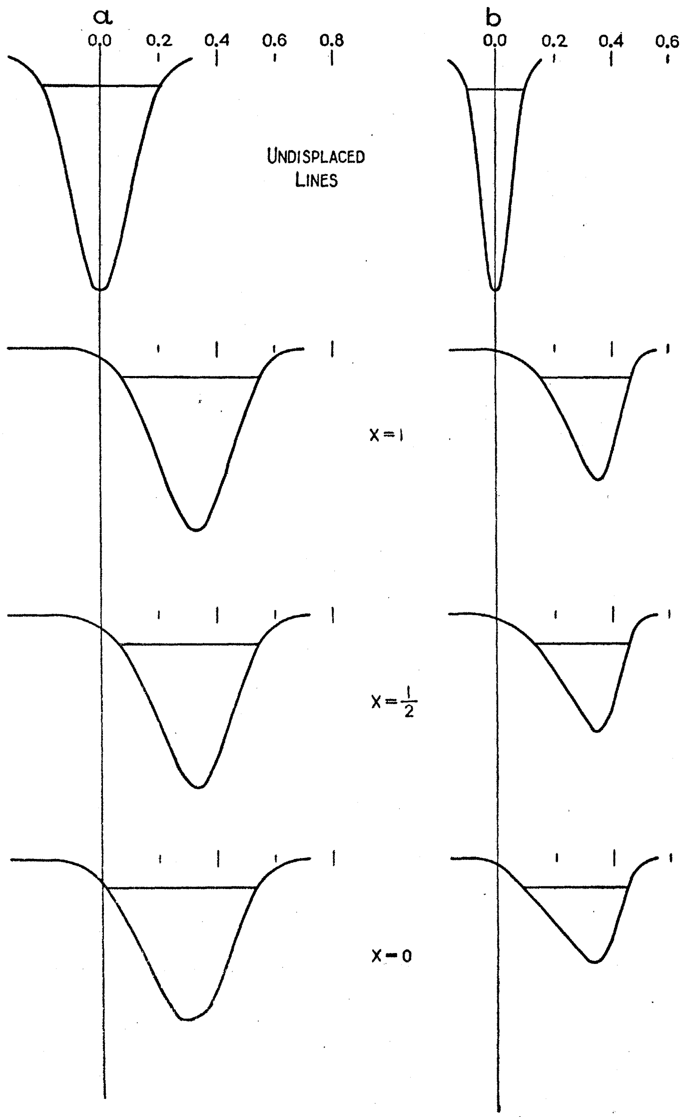

FIG. 3

There is considerable uncertainty in measures of the widths of absorption lines by ordinary methods, but it has been assumed that the measured widths lie between intensities of $\frac{1}{4} I'_0$, where I'_0 is the in-

tensity at maximum absorption of the undisplaced spectral line. It has also been assumed that the setting for the position of a line is such that the area under the intensity curve is divided equally and that the effect of absorption which is less than $\frac{1}{8} I'_0$ will not be appreciated; the results, however, are but slightly affected by the value assumed, provided it is within reasonable limits.

The lines of the spectrum of δ Cephei are much wider at minimum than at maximum light (and velocity).[4] Those shown in figures 3a and 3b are of the same width as observed in the case of δ Cephei at minimum and maximum, respectively, when a stellar spectograph of high dispersion and resolving power is used. The change observed in line width, according to results by Adams and by Young,[5] appears to have twice the period and about three times the amplitude of the variation which would be produced in a pulsating star by the distribution of velocity over the stellar disk; this latter variation, therefore, although it might be measurable if isolated, would probably be completely masked by the larger variation due to other causes.

Since practically all of the asymmetry is above the limit $\frac{1}{8} I'_0$ any deviation from a symmetrical form of the absorption lines would at all phases of the pulsation be inconspicuous on a spectrogram, and the lines would appear to be shifted bodily. The asymmetry would be most pronounced at maximum when the lines are comparatively narrow; but even then it seems doubtful if it could be observed if we are right in assuming that the lines for a Cepheid at maximum are best represented by curves such as those shown in figure 3, with values of x between $\frac{1}{2}$ and 1.

We conclude from these results that it is doubtful if either the periodic broadening or the slight asymmetry of the lines, due to a distribution of velocity over the stellar disk in accordance with the pulsation hypothesis, could be observed by any method thus far used.

[1] Shapley, Harlow, *Mt. Wilson Contr.*, No. 99, *Astrophys. J., Chicago, Ill.*, **41**, 1915 (291–306), pp. 292–295.

[2] Jeans, J. H., *Mon. Not. R. Astr. Soc., London*, 78, 1917 (28–36), p. 35.

[3] Wadsworth, F. L. O., *Phil. Mag., London*, Series V, **43**, 1897 (317–343).

[4] Adams, W. S., *Observatory, London*, **42**, 1919 (167–168).

[5] Young, R. K., *J. R. Astr. Soc. Canada, Toronto, Can.*, 13, 1919 (45–54).